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THE SOLUTION OF UNSTEADY CONTACT PROBLEMS IN THE PRESENCE OF COUPLING FORCES†

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The unsteady dynamic problem of a massive strip-shaped punch placed on a semi-bounded medium is considered. The punch is rigidly attached to the support, which is a laminate made of layers rigidly attached to one another. The lowest layer is rigidly attached to a non-deformable foundation or half-space. The punch is subject to an arbitrary load, the time dependence of which is specified. The proposed efficient method of solving problems of this kind makes it possible to study in detail the dynamics of the punch for various physical and geometric parameters of the supporting medium. The behaviour of the punch in the presence of coupling (adhesion) is compared with the behaviour when there is no friction in the contact domain. Numerical examples are considered. This study of unsteady stationary contact problems is a continuation of [1, 2].‡

1. FORMULATION OF THE PROBLEM

Suppose a rigid strip-shaped punch of width $2a$ with flat base rigidly attached to a semi-bounded layered medium occupying the domain $-\infty \leqslant x,\ y \leqslant \infty,\ z \leqslant 0$ is acted upon by a load of specified time dependence. The load, which is attached to the centre of mass of the punch, can be decomposed into a force $\mathbf{P}(t)=\{P_1(t),\ P_2(t)\}$ and moment $M(t)$. The displacements $u^0(t)=\{u_1^0,\ u_2^0\}$ of the points of the punch can be expressed in the form

$$u_1^0=u_1,\quad u_2^0=u_2+\varphi x$$

where u_1 and u_2 are the horizontal and vertical components of the displacement of the centre of mass of the punch, which coincides with the origin $(x=0,\ z=0)$ of the system of coordinates, and where φ is the angle of rotation about the centre of mass of the punch.

The problem can be reduced to the simultaneous solution of the equations of motion of the punch and the differential equations of motion of the medium under complete contact conditions

$$\mathbf{u}^0(t)=\mathbf{w}(x,0,t),\quad x\leqslant a$$

where $\mathbf{w}(x,\ z,\ t)=\{w_1,\ w_2\}$ are the displacements of the points of the medium.

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‡See also DOROKHOV I. V., PRYAKHINA A. D., and FREIGEIT M. R., The problem of time-dependent loads applied to a massive rigid punch on the surface of a multilayer support. Rostov-on-Don, 1992. Deposited at VINITI 10.07.92, No. 2266-B92.

Assuming that the system is initially at rest, we write the Laplace transforms of the equations of motion of the punch

$$
\begin{aligned}
&mp^2\mathbf{u}=\mathbf{P}(p)-\mathbf{Q}(p), \qquad \mathbf{Q}(p)=\int_{-a}^{a}\mathbf{q}(x)dx\\
&Jp^2\varphi=M(p)-R(p), \qquad R(p)=\int_{-a}^{a}q_2(x)x\,dx
\end{aligned}
\tag{1.1}
$$

Here m is the mass of the punch, J is the moment of inertia about the horizontal axis passing through the centre of mass of the punch, $\mathbf{q}(x)=\{q_1, q_2\}$ is the vector representing the shear and normal stresses under the punch, $\mathbf{Q}(p)$ and $R(p)$ are the resultant contact pressure and the moment of the normal contact stress component in the contact domain between the punch and the medium, and p is the Laplace transformation parameter.

On applying Fourier and Laplace integral transformations to the Lamé equations describing the motion of the medium and to the boundary conditions of the problem and taking the contact condition into account, it becomes necessary [3] to solve the following system of integral equations of the first kind with respect to the unknown contact stresses $\mathbf{q}(x)=\{q_1, q_2\}$

$$
\mathbf{Kq}=\int_{-a}^{a}\mathbf{k}(x-\xi,p)\mathbf{q}(\xi,p)d\xi=\mathbf{u}^0(p), \quad |x|\leqslant a \tag{1.2}
$$

$$
\mathbf{k}(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{K}(\alpha,pe^{-i\zeta})\,e^{-i\alpha x}d\alpha \tag{1.3}
$$

where α is the parameter of the Fourier transformation with respect to x.

Relationships (1.2) and (1.3) are fundamental in the study of unsteady interactions between a punch and a semi-bounded medium with visco-elastic properties described within the framework of the linear model of partially independent internal friction [1]. The coefficient of internal friction $\zeta=0$ corresponds to the case of an elastic medium, the integral (1.3) being taken along a contour σ determined by the radiation conditions [3]. The matrix-valued function $\mathbf{K}(\alpha, p)=\|K_{mn}\|_{m,n=1}^2$ is determined by the type of medium. For semi-bounded layered media it has the same form as in problems of steady oscillation with oscillation frequency ω changed to ip (i is the square root of minus one).

The properties of K_{mn} for semi-bounded media are described in detail in [3]. It should be noted that all functions K_{mn} have the same poles $\pm p_k$ ($k=1, 2, \ldots, n$) for all functions, K_{mn} being even and K_{mn} ($m\neq n$) odd functions of α with $K_{12}=-K_{21}=i\alpha L(\alpha)$. The functions K_{mn} have the following asymptotic representation as $|\alpha|\to\infty$

$$
K_{mn}=c|\alpha|^{-1}[1+O(\alpha^{-1})], \quad K_{12}=ib\alpha^{-1}[1+O(\alpha^{-1})], \quad c>|b| \tag{1.4}
$$

The above properties of the kernels ensure that the original system of equations (1.2) is uniquely solvable in $L_\lambda(-a, a)$, $\lambda>1$. Uniqueness criteria were established in [3].

2. CONSTRUCTION OF THE GOVERNING RELATIONSHIPS

Let $\mathbf{q}_0(x, \eta, A_1, A_2)$ be the solution of the system of equations (1.2) with known right-hand side

$$
\mathbf{Kq}=\mathbf{A}e^{-i\eta x}, \quad \mathbf{A}=\{A_1,A_2\}
$$

Then, by the linearity of the problem, the solution of the contact problem (1.2), (1.3) will be given by the relation

$$\mathbf{q}(x) = u_1\mathbf{q}^1 + u_2\mathbf{q}^2 + \varphi\mathbf{q}^3 \tag{2.1}$$

where $\mathbf{q}^k$ are the solutions of the contact problems

$$\mathbf{K}\mathbf{q}^1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{K}\mathbf{q}^2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad \mathbf{K}\mathbf{q}^3 = \begin{pmatrix}0\\x\end{pmatrix}$$

connected with $\mathbf{q}_0$ as follows:

$$\mathbf{q}^1 = \mathbf{q}_0(x,0,1,0), \quad \mathbf{q}^2 = \mathbf{q}_0(x,0,0,1), \quad \mathbf{q}^3 = i\frac{\partial \mathbf{q}_0}{\partial \eta}(x,0,0,1)$$

The equations of motion (1.1) take the form

$$mp^2u_1 = P_1 - Q_1^1u_1 - Q_1^2u_2 - Q_1^3\varphi$$

$$mp^2u_2 = P_2 - Q_2^1u_1 - Q_2^2u_2 - Q_2^3\varphi$$

$$Jp^2\varphi = M - R^1u_1 - R^2u_2 - R^3\varphi$$

$$\mathbf{Q}^k = \int_{-a}^{a} \mathbf{q}^k(x)dx, \quad R^k = \int_{-a}^{a} q_2^k(x)x\,dx, \quad k = 1,2,3$$

Since the x-axis is the axis of symmetry of the problem, we have

$$(mp^2 + Q_1^1)u_1 + Q_1^3\varphi = P_1, \quad (mp^2 + Q_2^2)u_2 = P_2, \quad (Jp^2 + R^3)\varphi + R^1u_1 = M$$

Using the reciprocity theorem, it can be shown that $Q_1^3 \equiv R^1$. Then the displacement components of the punch can be written as follows:

$$\begin{aligned} u_1 &= [P_1(Jp^2 + R^3) - MQ_1^3]\Delta_0^{-1} \\ u_2 &= P_2(mp^2 + Q_2^2)^{-1} \\ \varphi &= [M(mp^2 + Q_1^1) - P_1Q_1^3]\Delta_0^{-1} \\ \Delta_0 &= (mp^2 + Q_1^1)(Jp^2 + R^3) - (Q_1^3)^2 \end{aligned} \tag{2.2}$$

The four functionals Q_1^1, Q_1^3, Q_2^2, R^3 must obviously be determined in order to construct a solution.

3. CONSTRUCTION OF THE SOLUTION OF THE CONTACT PROBLEM BY THE FICTITIOUS DISSIPATION METHOD

Consider the system of integral equations (1.2)

$$K_{11}q_1^0 + K_{12}q_2^0 = A_1e^{-i\eta x}, \quad K_{21}q_1^0 + K_{22}q_2^0 = A_2e^{-i\eta x} \tag{3.1}$$

where $\mathbf{q}_0(x) = \{q_1^0,\ q_2^0\} \equiv \mathbf{q}_0(x,\ \eta,\ A_1,\ A_2)$.

We will represent the matrix $\mathbf{K}(\alpha) \equiv \mathbf{K}(\alpha,\ p)$ as a product $\mathbf{K}(\alpha) = \mathbf{S}(\alpha)\Pi(\alpha)$ and construct $\mathbf{S}(\alpha)$ in such a way that its matrix elements have no singularities on the real axis and preserve the behaviour of $\mathbf{K}(\alpha)$ (of the form (1.4)) at infinity. We choose

$$\mathbf{S}(\alpha)=\begin{Vmatrix} s_1 & is_2 \\ -is_2 & s_1 \end{Vmatrix}\frac{\beta^2}{(B^2+\alpha^2)^{1/2}} \tag{3.2}$$

where

$$s_1=\operatorname{ch}(2\delta\psi),\quad s_2=\operatorname{sh}(2\delta\psi),\quad \beta^4=c^2-b^2$$

$$\psi=\operatorname{arctg}(\alpha/B),\quad \delta=\pi^{-1}\operatorname{arcth}(b/c)$$

For the media under consideration $c=1-\nu$ and $b=(1-2\nu)/2$, where ν is Poisson's ratio of the upper layer. In this case the matrix elements of $\Pi(\alpha)=\mathbf{S}^{-1}\mathbf{K}$ retain all the singularities of $\mathbf{K}(\alpha)$ and have the form

$$\Pi_{mm}=(\alpha^2+B^2)^{1/2}\beta^{-2}[K_{mm}s_1-\alpha L s_2],\quad m=1,2$$

$$\Pi_{mn}=(\alpha^2+B^2)^{1/2}\beta^{-2}[L s_1-K_{nn}s_2\alpha^{-1}],\quad m\neq n$$

We note that $\Pi_{mn}(\alpha)$ are even functions at infinity and $\Pi(\alpha)$ reduces to the identity matrix

$$\mathbf{\Pi}(\alpha)=\begin{Vmatrix} \Pi_{11} & i\alpha\Pi_{12} \\ -i\alpha\Pi_{21} & \Pi_{22} \end{Vmatrix}\underset{\alpha\to+\infty}{\sim}\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

Using the fictitious dissipation method, we seek a solution in the form

$$\mathbf{q}_0(x)=\mathbf{q}_*(x)+\varphi(x) \tag{3.3}$$

such that the relationships

$$Q_*(\pm p_k)=\int_{-a}^{a} q_*(x)e^{\pm p_k x}dx=0,\quad k=1,\dots,n$$

are satisfied.

As the components of $\varphi(x)$ we take systems of δ-functions with disjoint supports at $x_k=\pm y_k$, where the points y_k divide the interval $(0, a)$ into equal subintervals

$$\varphi(x)=\sum_{k=1}^{2n}\mathbf{C}_k\delta(x-x_k) \tag{3.4}$$

$\mathbf{C}_k=\{C_k^1,\ C_k^2\}$ being constants to be determined.

We introduce a new unknown vector-valued function $\mathbf{t}(x)$ by

$$\mathbf{t}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{T}(\alpha)e^{-i\alpha x}d\alpha,\quad \mathbf{T}(\alpha)=\mathbf{\Pi}(\alpha)\mathbf{Q}_*(\alpha) \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.1) and taking (3.5) into account, we arrive at the following system of integral equations

$$\mathbf{St}\equiv\int_{-a}^{a} s(x-\xi)\mathbf{t}(\xi)d\xi=\mathbf{A}e^{-i\eta x}-\sum_{k=1}^{2n}s(x-x_k)\mathbf{C}_k$$

$$s(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{S}(\alpha)e^{-i\alpha x}d\alpha \tag{3.6}$$

The matrix elements of $\mathbf{S}(\alpha)$ are given by (3.2). Now the operator on the left-hand side of (3.6) involves strong damping, the matrix-valued kernel $\mathbf{S}$ under the integral sign has no singularities on the real axis, and it turns out that various methods for solving static problems can be used to determine $\mathbf{t}(x)$. The methods of constructing the inverse operator $\mathbf{S}^{-1}$, which depend on properties of $\mathbf{S}(\alpha)$, are described in [3–6].

Let $\mathbf{t}_0(x)=\chi(x,\ \eta)\mathbf{A}$ be the solution of (3.6) with right-hand side $\mathbf{A}e^{-i\eta x}$, that is, let $\mathbf{T}_0(\alpha)=\mathbf{X}(\alpha,\ \eta)\mathbf{A}$ in terms of Fourier–Laplace transforms. Then, due to the linearity of the problem, the solution of (3.6) can be written in the form

$$\mathbf{T}(\alpha)=\mathbf{T}_0(\alpha)-\sum_{k=1}^{2n}[\mathbf{I}e^{i\alpha x_k}+\mathbf{Z}(\alpha,x_k)]\mathbf{C}_k \tag{3.7}$$

$$\mathbf{t}(x)=\mathbf{t}_0(x)-\varphi(x)-\frac{1}{2\pi}\sum_{k=1}^{2n}\int_\sigma Z(\alpha,x_k)\mathbf{C}_k e^{-i\alpha x}d\alpha \tag{3.8}$$

$$\mathbf{Z}(\alpha,x)=\frac{1}{2\pi}\int_\sigma \mathbf{S}^{-1}(\alpha)[\mathbf{K}(\eta)-\mathbf{S}(\eta)]\mathbf{S}(\alpha)\mathbf{X}(\alpha,\eta)e^{-i\eta x}d\eta \tag{3.9}$$

From (3.5) we find that

$$\mathbf{q}_*(x)=\int_{-a}^{a}\mathbf{Q}_*(\alpha)e^{-i\alpha x}d\alpha,\quad \mathbf{Q}_*(\alpha)=\mathbf{\Pi}^{-1}(\alpha)\mathbf{T}(\alpha) \tag{3.10}$$

By the fictitious dissipation method, in order that $\mathbf{q}_*(x)$ be in $\mathbf{L}_\lambda$ and have support only in $[-a,\ a]$, the relationships

$$\mathbf{T}(\pm\xi_k)=0,\quad k=1,\ldots,n \tag{3.11}$$

must be satisfied, $\pm\xi_k$ being the poles of the inverse matrix $\Pi^{-1}(\alpha)$, the same for all elements Π_{ij}^{-1}. The relationships (3.11) constitute an algebraic system of $4n$ equations from which to determine the $4n$ unknowns $C_k^1,\ C_k^2\ (k=1,\ \ldots,2n)$.

Using (3.3), (3.4), (3.10), (3.7) and (3.8), we obtain the desired solution of the contact problem in the form

$$\begin{aligned}\mathbf{q}_0(x)=&\left\{\chi(x,\eta)+\frac{1}{2\pi}\int_\sigma(\mathbf{\Pi}^{-1}(\alpha)-\mathbf{I})\mathbf{X}(\alpha,\eta)e^{-i\alpha x}d\alpha\right\}\mathbf{A}-\\&-\frac{1}{2\pi}\sum_{k=1}^{2n}\left[\int_\sigma\mathbf{\Pi}^{-1}(\alpha)\mathbf{Z}(\alpha,x_k)e^{-i\alpha x}d\alpha+\int_\sigma(\mathbf{\Pi}^{-1}(\alpha)-\mathbf{I})e^{-i\alpha(x-x_k)}d\alpha\right]\mathbf{C}_k\end{aligned} \tag{3.12}$$

and, correspondingly

$$\mathbf{Q}_0(\alpha)=\mathbf{\Pi}^{-1}(\alpha)\mathbf{T}(\alpha)+\sum_{k=1}^{2n}\mathbf{C}_k e^{i\alpha x_k} \tag{3.13}$$

To construct $\mathbf{t}_0(x)$ we use the factorization method, which enables us to obtain a very simple representation of the solution (3.6) as a degenerate component and a boundary layer.

A commutative factorization of the matrix-valued function $\mathbf{S}(\alpha)$ can be realized explicitly according to the general theorems [7]. In this case the contour σ coincides with the real axis and

$$\mathbf{S}(\alpha)=\mathbf{S}_+(\alpha)\mathbf{S}_-(\alpha)=\mathbf{S}_-(\alpha)\mathbf{S}_+(\alpha)$$

$$\mathbf{S}_\pm(\alpha)=\beta/2(B\mp i\alpha)^{-1/2}\begin{Vmatrix} s_1^\pm & is_2^\pm \\ -is_2^\pm & s_1^\pm\end{Vmatrix}$$

$$s_1^{\pm} = (B \mp i\alpha)^{\pm i\delta} + (B \mp i\alpha)^{\mp i\delta}; \quad s_2^{\pm} = (B \mp i\alpha)^{\pm i\delta} - (B \mp i\alpha)^{\mp i\delta}$$

Using the factorization method, one can reduce the solution of the system of integral equations (3.6) to the solution of a system of equations of the second kind with a completely continuous operator of the form

$$\begin{aligned} \mathbf{X}(z) &= -\frac{1}{2\pi i}\int_\sigma \mathbf{S}_+^{-1}(\alpha)[\mathbf{S}_-(\alpha)\mathbf{Y}(\alpha)e^{-2i\alpha a} + \mathbf{F}(\alpha)e^{-ia\alpha}]\frac{d\alpha}{\alpha+z} \\ \mathbf{Y}(z) &= -\frac{1}{2\pi i}\int_\sigma \mathbf{S}_-^{-1}(\alpha)[\mathbf{S}_+(-\alpha)\mathbf{X}(\alpha)e^{-2ia\alpha} + \mathbf{F}(-\alpha)e^{-ia\alpha}]\frac{d\alpha}{\alpha+z} \end{aligned} \tag{3.14}$$

$$\mathbf{F} = \{F_1, F_2\}, \qquad F_k(\alpha) = 2\pi A_k\delta(\alpha-\eta)$$

$$\mathbf{X} = \{X_1, X_2\}, \qquad \mathbf{Y} = \{Y_1, Y_2\}$$

The vector-valued functions $X(\alpha)$ and $Y(\alpha)$ are regular below σ and decay at least exponentially there [8].

The vector-valued function $\mathbf{t}_0(x)$ has the form

$$\mathbf{t}_0(x) = \frac{1}{2\pi}\int_\sigma [\mathbf{S}^{-1}(\alpha)\mathbf{F}(\alpha) + \mathbf{S}_-^{-1}(\alpha)\mathbf{X}(-\alpha)e^{ia\alpha} + \mathbf{S}_+^{-1}(\alpha)\mathbf{Y}(\alpha)e^{-ia\alpha}]e^{-i\alpha x}d\alpha \tag{3.15}$$

Neglecting small integral terms, one can represent the approximate solution of the system of integral equations (3.14) in the form

$$\mathbf{X}(z) = i\mathbf{S}_+^{-1}(\eta)\mathbf{A}\frac{e^{-i\eta a}}{\eta+z}; \quad \mathbf{Y}(z) = i\mathbf{S}_-^{-1}(\eta)\mathbf{A}\frac{e^{i\eta a}}{z-\eta} \tag{3.16}$$

Substituting (3.16) into (3.15), we obtain the representation

$$\mathbf{t}_0(x) = \boldsymbol{\chi}(x,\eta)\mathbf{A}$$

$$\boldsymbol{\chi}(x,\eta) = \{-[\mathbf{G}(x,\eta) + \mathbf{G}^T(-x,-\eta)](B^2+\eta^2)^{1/2}(2\beta^2)^{-1} + \mathbf{S}^{-1}(\eta)\}e^{-i\eta x}$$

$$\mathbf{G}(x,\eta) = \begin{Vmatrix} G_1 & -iG_2 \\ iG_2 & G_1 \end{Vmatrix}$$

$$G_{1,2} = \Gamma^{-1}(\nu_2)\Gamma[\nu_2,(B+i\eta)(a-x)]e^{2\delta\psi_0} \pm \Gamma^{-1}(\nu_1)\Gamma[\nu_1,(B+i\eta)(a-x)]e^{-2\delta\psi_0}$$

$$\nu_{1,2} = -1/2 \mp i\delta, \quad \psi_0 = \mathrm{arctg}(\eta/B)$$

The expression for $G_{1,2}$ can be expressed as

$$G_{1,2} = 2s_{1,2}(\eta) - w_{1,2}(x,\eta) - \sigma_{1,2}(x,\eta) \tag{3.17}$$

$$w_{1,2}(x,\eta) = e^{2\delta\psi_0}\pi_\gamma(\nu_2,\eta,x) \pm e^{-2\delta\psi_0}\pi_\gamma(\nu_1,\eta,x)$$

$$\sigma_{1,2}(x,\eta) = e^{2\delta\psi_0}\theta(\nu_2,\eta,x) \pm e^{-2\delta\psi_0}\theta(\nu_1,\eta,x)$$

$$\pi_\gamma(\varepsilon,\eta,x) = \Gamma^{-1}(\varepsilon+1)\gamma[\varepsilon+1,(B+i\eta)(a-x)]\theta(\varepsilon,\eta,x) =$$

$$= \Gamma^{-1}(\varepsilon+1)e^{-(B+i\eta)(a-x)}[(B+i\eta)(a-x)]^\varepsilon$$

Here $\Gamma(x)$, $\gamma(\varepsilon, x)$ and $\Gamma(\varepsilon, x)$ are the complete and incomplete Euler gamma-function, respectively, and transposition is denoted by the index T.

It can be seen from (3.17) that in the contact problem with coupling the solution contains an oscillatory singularity of the form $(a \pm x)^{-1/2 \mp i\delta}$ on the edge of the punch.

The Fourier transform $\mathbf{T}_0(\alpha)$ of the vector-valued function $\mathbf{t}_0(x)$ in the interval $(-a, a)$ has the form

$$\mathbf{T}_0(\alpha) = \mathbf{X}(\alpha, \eta)\mathbf{A}$$

$$\mathbf{X}(\alpha, \eta) = [2i\beta^2(\alpha - \eta)]^{-1}\{e^{ia(\alpha-\eta)}\mathbf{R}(\alpha, \eta) - e^{-ia(\alpha-\eta)}\mathbf{R}^T(-\alpha, -\eta)\}$$

$$\mathbf{R}(\alpha, \eta) = \begin{Vmatrix} r_1 & ir_2 \\ -ir_2 & r_1 \end{Vmatrix}$$

$$r_{1,2} = (B + i\alpha)^{-\nu_1}(B - i\eta)^{-\nu_2}\pi(\nu_1, \alpha) \pm (B + i\alpha)^{-\nu_2}(B - \eta)^{-\nu_1}\pi(\nu_2, \alpha) \pm$$

$$\pm (B^2 + \eta^2)^{1/2}[e^{2\delta\psi_0}\pi(\nu_1, -\eta) \pm e^{-2\delta\psi_0}\pi(\nu_2, -\eta) - 2s_{1,2}]$$

$$\pi(\varepsilon, \alpha) = \gamma[\varepsilon + 1, 2a(B + i\alpha)]\Gamma^{-1}(\varepsilon + 1) = \pi_\gamma(\varepsilon, \alpha, -a)$$

We can obtain the final formulae from which to compute the shear and normal stresses under the punch by substituting the expressions for $\mathbf{t}_0(x)$, $\chi(x, \eta)$ and $\mathbf{X}(\alpha, \eta)$ into the integral representation (3.12) of the solution. After multiplying the matrices and some reduction one can evaluate the integrals (3.9) by residues, since the integrands decay exponentially in the lower half-plane and have no branching points there. The remaining integrals can be evaluated from the formulae of the operational calculus.

Omitting the computations, we shall present the general form of the approximate solution of the system of equations

$$\mathbf{q}_0(x) = \Big\{-2\beta^2\mathbf{K}^{-1}(\eta) + (B^2 + \eta^2)^{1/2}[\mathbf{V}(x, \eta) + \mathbf{V}^*(-x, -\eta)] +$$

$$+ e^{-i\eta(a-x)}(B - i\eta)\sum_{l=1}^{n}\mathbf{M}(\xi_l, x, \eta) + e^{i\eta(a+x)}(B + i\eta)\sum_{l=1}^{n}\mathbf{M}^*(\xi_l, -x, -\eta)\Big\}\mathbf{A}e^{-i\eta x}(2\beta^2)^{-1} +$$

$$+ \frac{i}{2}\sum_{k=1}^{2n}\left\{\sum_{j=1}^{n}\left(\frac{B + ip_j}{B - ip_j}\right)^{1/2}[\mathbf{N}(p_j, x_k, x) + \mathbf{N}^*(p_j, -x_k, -x)] +\right.$$

$$\left.+ \sum_{j=1}^{n}\sum_{l=1}^{n}[\mathbf{Y}(\xi_l, p_j, x_k, x) + \mathbf{Y}^*(\xi_l, p_j, -x_k, -x)]\right\}\mathbf{C}_k$$

$$\mathbf{V}(x, \eta) = \mathbf{\Pi}^{-1}(\eta)\mathbf{W}(x, \eta) + \boldsymbol{\sigma}(x, \eta)$$

$$\mathbf{M}(\xi, x, \eta) = \mathbf{B}(-\xi)\mathbf{E}^1(-\xi, x, \eta)e^{-i\xi(a-x)}(\eta + \xi)^{-1} - \mathbf{B}(\xi)\mathbf{E}(\xi, x, \eta)e^{i\xi(a-x)}(\eta - \xi)^{-1}$$

$$\mathbf{N}(\kappa, y, x) = e^{i\kappa(a-y) + i\kappa(a-x)}\mathbf{A}(-\kappa)\boldsymbol{\sigma}(x, \kappa)$$

$$\mathbf{E}^1(\xi, x, \eta) = \mathbf{b}(\xi, \eta) - \mathbf{E}(\xi, x, \eta)$$

$$\mathbf{Y}(\xi, \kappa, y, x) = e^{i\kappa(a-y)}[\mathbf{B}(-\xi)\mathbf{A}(-\kappa)\mathbf{\Xi}^{-1}(-\xi, x, \kappa)e^{-i\xi(a-x)}(\xi - \kappa)^{-1} -$$

$$- \mathbf{B}(\xi)\mathbf{A}(-\kappa)\mathbf{E}(\xi, x, \kappa)e^{i\xi(a-x)}(-\kappa - \xi)^{-1}]$$

$$\mathbf{A}(p_j) = \operatorname*{Res}_{\alpha = p_j}\mathbf{\Pi}(\alpha), \quad \mathbf{B}(\xi_j) = \operatorname*{Res}_{\alpha = \xi_j}\mathbf{\Pi}^{-1}(\alpha)$$

$$\mathbf{W}(x, \eta) = \begin{Vmatrix} w_1 & -iw_2 \\ iw_2 & w_1 \end{Vmatrix}, \quad \boldsymbol{\sigma}(x, \eta) = \begin{Vmatrix} \sigma_1 & -i\sigma_2 \\ i\sigma_2 & \sigma_1 \end{Vmatrix}$$

$$\mathbf{E}(\xi, x, \kappa) = \begin{Vmatrix} e_1 & -ie_2 \\ ie_2 & e_1 \end{Vmatrix}, \quad \mathbf{b}(\alpha, \eta) = \begin{Vmatrix} b_1 & -ib_2 \\ ib_2 & b_1 \end{Vmatrix}$$

$$e_{1,2}(\xi, x, \eta) = \tau(\nu_2, \xi, \eta)\pi_\gamma(\nu_2, \xi, x) \pm \tau(\nu_1, \xi, \eta)\pi_\gamma(\nu_1, \xi, x)$$

$$\tau(\varepsilon,\xi,\eta)=\left(\frac{B+i\xi}{B-i\eta}\right)^{-\varepsilon}, \quad b_{1,2}(\alpha,\eta)=\tau(\nu_2,\alpha,\eta)\pm\tau(\nu_1,\alpha,\eta)$$

System (3.11) from which to determine C_k takes the form

$$\sum_{k=1}^{2n}\mathbf{f}(\pm\xi_l,x_k)\mathbf{C}_k=\mathbf{X}(\pm\xi_l,\eta)\mathbf{A}, \quad l=1,\ldots,n$$

The Fourier transform $\mathbf{Q}(\alpha,\eta)$ given by (3.13) can be determined from the simple formula

$$\mathbf{Q}_0(\alpha)=\mathbf{\Pi}^{-1}(\alpha)\left\{\mathbf{X}(\alpha,\eta)\mathbf{A}-\sum_{k=1}^{2n}\mathbf{f}(\alpha,x_k)\mathbf{C}_k\right\} \tag{3.18}$$

$$\mathbf{f}(\alpha,x)=[e^{i\alpha a}\mathbf{F}(\alpha,x)+e^{-i\alpha a}\mathbf{F}^*(-\alpha,-x)]/2$$

$$\mathbf{F}(\alpha,x)=\sum_{j=1}^{n}\frac{e^{ip_j(\alpha-x)}}{2p_j(p_j+\alpha)}\mathbf{A}(-p_j)\mathbf{b}(\alpha,p_j)$$

The asterisk means that the skew-diagonal matrix elements depending on $(-x)$ must be multiplied by (-1).

4. DETERMINATION OF THE DISPLACEMENTS OF THE POINTS OF THE PUNCH

It is obvious that the functionals Q_1^1, Q_1^3, Q_2^2, R^3 are connected with the constructed solution $\mathbf{Q}_0(\alpha)\equiv\mathbf{Q}_0(\alpha,\eta,A_1,A_2)$ given by (3.18) by the following relationships

$$Q_1^1=Q_1^0(0,0,1,0);\quad Q_2^2=Q_2^0(0,0,0,1);\quad R^3=\frac{\partial^2 Q_2^0}{\partial\alpha\,\partial\eta}(0,0,0,1)$$

$$Q_1^3=i\frac{\partial Q_1^0}{\partial\eta}(0,0,0,1)=-i\frac{\partial Q_2^0}{\partial\alpha}(0,0,1,0)$$

The unknown functionals in (2.2) are therefore determined and, applying an inverse Laplace transformation, we obtain the displacements of the centre of mass and the rotation angle of the punch

$$\mathbf{u}(t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\mathbf{u}(p)e^{pt}dt \qquad \varphi(t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\varphi(p)e^{pt}dt, \quad \varepsilon>0$$

The physical conditions of the problem imply that the integrand has no roots in the right half-plane $\operatorname{Re}p>0$ and the integral along a straight line parallel to the imaginary axis can be replaced by the integral along the imaginary axis. Substituting $p=-i\omega$, the integral of the inverse Laplace transform can therefore be reduced to the Fourier integral

$$\mathbf{u}(t)=-\frac{2}{\pi}\int_0^\infty\operatorname{Im}\mathbf{u}(i\omega)\sin\omega t\,d\omega, \quad \varphi(t)=-\frac{2}{\pi}\int_0^\infty\operatorname{Im}\varphi(i\omega)\sin\omega t\,d\omega \tag{4.1}$$

These integrals can be computed using Filon's method [8].

To compute the reaction $\mathbf{Q}(t)$ of the base, the torque $R(t)$, and the contact stresses $\mathbf{q}(x,t)$ it is necessary to replace the integrand in (4.1) by $\mathbf{Q}(p)$, $R(p)$, or $\mathbf{q}(x,p)$, respectively. Then $\mathbf{q}(x,t)$ can be determined from (2.1), and

$$Q_1(p)=u_1Q_1^1+\varphi Q_1^3, \quad Q_2(p)=u_2Q_2^2$$

$$R(p)=u_1Q_1^3+\varphi R^3=u_1R^1+\varphi R^3$$

5. THE INFLUENCE FUNCTION FOR A MULTILAYER MEDIUM IN A HALF-SPACE

A method of constructing the influence function $\mathbf{K}(\alpha, \beta, z, \omega)$ for a multilayer medium rigidly attached to a non-deformable base in three dimensions has been proposed in [2]. An advantage of this method is that it makes it possible to study unsteady problems with any number of layers. The Fourier transform of the displacement vector $\mathbf{w}(z) = \{w_1, w_2, w_3\}$ of the points of the medium can be obtained in the form (α and β are the parameters of the transform)

$$\mathbf{w}(z) = \mathbf{K}(\alpha,\beta,z,\omega)\mathbf{q}_0, \quad \mathbf{K}(\alpha,\beta,z,\omega) = (-1)^{k-1}(\mathbf{A}(z_k) - \mathbf{B}(z_k)\mathbf{F}_k^{-1}\mathbf{A}(-h_k))\times$$
$$\times \Pi_{i=k-1}^{1}\mathbf{F}_i^{-1}\mathbf{A}(-h_i)/\mu_k$$
$$\mathbf{F}_N = \mathbf{B}(-h_N), \quad \mathbf{F}_k = \mathbf{B}(-h_k) - g_k\mathbf{A}(-h_{k+1}) + g_k\mathbf{B}(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{A}(h_{k+1}), \quad k = 1,2,\ldots,N-1$$
$$z = z_k - 2\sum_{i=1}^{k} h_i + h_k, \quad k = 1,2,\ldots,N.$$

where h_k and μ_k are the half-thickness and the Lamé parameter of the kth layer, $g_k = \mu_k/\mu_{k+1}$, and $\mathbf{q}_0 = \{q_1, q_2, q_3\}$ is the surface load. The matrices $\mathbf{B}(z)$ and $\mathbf{A}(z)$ are given in [2].

We can obtain the solution for a medium rigidly attached to an elastic half-space by letting the thickness of the lowest layer tend to ∞. By changing the system of coordinates to $z^* = z_N - h_N$ in the lowest layer and taking the limit, we get

$$\mathbf{F}_N = 0, \quad \mathbf{F}_{N-1} = \mathbf{B}(-h_{N-1}) - g_{N-1}\mathbf{A}^\infty(0)$$
$$\mathbf{F}_k = \mathbf{B}(-h_k) - g_k\mathbf{A}(h_{k+1}) + g_k\mathbf{B}(h_{k+1})\mathbf{F}_{k+1}^{-1}\mathbf{A}(-h_{k+1}), \quad k = 1,2,\ldots,N-2$$
$$z = z_k - 2\Sigma_{i=1}^{k} h_i + h_k, \quad k = 1,2,\ldots,N-1$$
$$z = z^* - 2\Sigma_{i=1}^{N-1} h_i, \quad k = N$$
$$\mathbf{A}^\infty(z) = \begin{Vmatrix} \alpha^2 M + \beta^2 L & \alpha\beta[M-L] & -i\alpha N_1 \\ \alpha\beta[M-L] & \beta^2 M + \alpha^2 L & -i\beta N_1 \\ i\alpha N_2 & i\beta N_2 & R \end{Vmatrix}$$
$$M = \frac{2\sigma_2}{\lambda^2\Delta^\infty}[-\lambda^2 f(\sigma_1 z) + \gamma f(\sigma_2 z)], \quad L = \frac{4}{\sigma_2\lambda^2} f(\sigma_2 z)$$
$$N_{1,2} = \frac{2}{\Delta^\infty}[\gamma f(\sigma_{1,2} z) - \sigma_1\sigma_2 f(\sigma_{2,1} z)]$$
$$R = \frac{2\sigma_1}{\Delta^\infty}[-\lambda^2 f(\sigma_2 z) + \gamma f(\sigma_1 z)], \quad f(z) = \mathrm{ch}(z) + \mathrm{sh}(z) = e^z$$
$$\Delta^\infty = [\gamma^2 - \lambda^2\sigma_1\sigma_2], \quad \lambda^2 = \alpha^2 + \beta^2, \quad \gamma = \lambda^2 - \tfrac{1}{2}\theta_2^2$$
$$\sigma_i^2 = \lambda^2 - \theta_i^2, \quad \theta_1^2 = \varepsilon_N\theta_2^2, \quad \theta_2^2 = \rho_N\omega^2/\mu_N, \quad \varepsilon_N = (1-2\nu_N)/(2-2\nu_N)$$

In particular, for a layer rigidly attached to the half-space ($-\infty \leqslant x, \ y \leqslant \infty, \ z \leqslant 0$), we obtain the following displacements:
in the layer

$$\mathbf{w}(z) = [\mathbf{A}(z+h_1) - \mathbf{B}(z+h_1)\mathbf{F}_1^{-1}\mathbf{A}(-h_1)]\mathbf{q}_0\mu_1^{-1}$$

in the half-space

$$\mathbf{w}(z) = -\mathbf{A}^\infty(z+2h_1)\mathbf{F}_1^{-1}\mathbf{A}(-h_1)\mathbf{q}_0\mu_2^{-1}, \quad \mathbf{F}_1 = \mathbf{B}(-h_1) - g_1\mathbf{A}^\infty(0)$$

6. NUMERICAL ANALYSIS OF THE PROBLEM

A numerical analysis was carried out for a strip-shaped punch rigidly attached to a multilayer medium.

Figure 1 shows the effect of coupling forces within the contact domain on the behaviour of a punch in contact with a layer of thickness $H = 0.5$ and subjected to a force $P_2(t) = te^{-2.5t}$. The solid line corresponds to the problem with coupling (adhesion) within the contact domain, while the dashed line corresponds to the problem without friction. It can be seen from Fig. 1 that coupling forces result in a slightly smaller maximum displacement of the punch as compared with the problem without friction.

For systems in which the thickness of the layer is comparable with the dimensions of the punch the effect of coupling forces within the contact domain on the vertical displacements of the punch turns out to be small. We observe that the basic patterns in the behaviour of the punch remain the same for media with more complex properties (layered ones).

The time dependence of the angle of rotation and the horizontal displacement of the punch subjected to the moment $M(t) = H(t) - H(t-2)$ is shown in Fig. 2. Curve 1 corresponds to the horizontal displacement and curve 2 to the angle of rotation of the punch. In the case when the punch is acted upon by a horizontal load the behaviour of u_1 and φ is similar to that shown in Fig. 2, but, the horizontal displacement of the punch predominates. All quantities are presented in dimensionless form: the displacements relative to the half-thickness a of the punch, the load to the rigidity μ of the layer, and the time to $(\rho/\mu)^{1/2}a$, Moreover, $\nu = 0.3$, $\zeta = 0.2$ is the coefficient of friction of the medium, and $M = 1$ is the mass of the punch.

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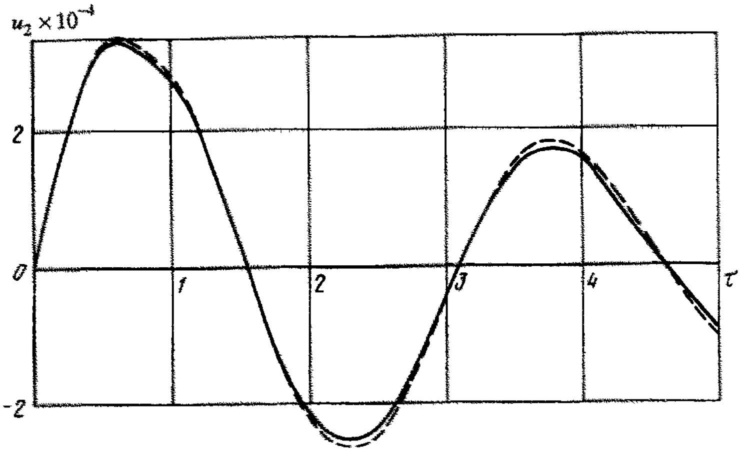

Fig. 1.

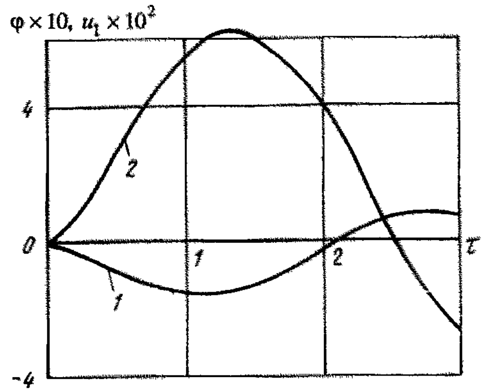

Fig. 2.

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